

# Energy flow from a battery to other circuit elements: Role of surface charges

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A qualitative description of energy transfer from a battery to a resistor using the Poynting vector was recently published. We make this argument quantitative by considering a long current carrying wire and showing that the energy transferred across a plane perpendicular to the wire is equal to the Joule heating in the wire beyond this plane. © 2010 American Association of Physics Teachers.

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In an electromagnetic field of electric field strength  $\mathbf{E}$  and magnetic field strength  $\mathbf{B}$ , the energy carried by the field is given by the Poynting vector<sup>1-6</sup>

$$\mathbf{S} = \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B}. \quad (1)$$

A common example of energy transport by the Poynting vector is its application to a current carrying wire.<sup>2,3</sup> Consider a wire of length  $L$  and radius  $a$  along the  $z$ -axis carrying a current  $I$  in the negative  $z$ -direction distributed uniformly over its cross-section. The voltage drop across the wire is  $V$ . The conductivity of the wire is  $\sigma$ . The electric field inside the wire (we use cylindrical coordinates with the unit vectors<sup>2</sup>  $\hat{\phi}$ , and  $\hat{z}$ ) is

$$\mathbf{E} = -\frac{V}{L} \hat{z} = \frac{\mathbf{j}}{\sigma} = -\frac{I}{\pi a^2 \sigma} \hat{z}. \quad (2)$$

Inside the wire, there is also a magnetic field given by

$$\mathbf{B} = -\frac{\mu_0 I s}{2\pi a^2} \hat{\phi} \quad (3)$$

for  $s \leq a$ , where  $s$  is the distance from the axis of the wire. Therefore the Poynting vector is

$$\mathbf{S} = -\frac{I^2 s}{2\pi^2 a^4 \sigma} \hat{s}. \quad (4)$$

Thus, the energy  $W$  flowing into the cylinder of radius  $s$  and length  $L$  through its surface is

$$W = \int \mathbf{S} \cdot d\mathbf{a} = \frac{I^2}{\pi a^2 \sigma} \left(\frac{s}{a}\right)^2 L = \sigma \left(\frac{V}{L}\right)^2 \pi s^2 L, \quad (5)$$

which is equal to the Joule heating in the cylinder. For  $s=a$ , Eq. (5) gives the amount of thermal energy generated in the wire due to Joule heating. The picture that emerges from these considerations is that the electromagnetic field around a current carrying wire is such that the energy dissipated in the wire is brought into it by the corresponding Poynting vector through each point of its surface. For a wire connected to a battery, the Poynting vectors given by Eq. (4) are shown in Fig. 1 at four sections of the wire. The question arises: How does the energy flow from the battery to the section of the wire where it is dissipated? Does it go out of the battery in all directions and then return back to the wire or does it

flow along the wire? This issue is not addressed in most textbooks on electricity and magnetism.

The answer to this question is provided by considering the surface charges<sup>4</sup> that give rise to a radial component of the electric field outside the wire. The problem has been discussed in the context of a ring battery driving a current through a circuit consisting of a thin cylindrical shell and a resistor on its axis.<sup>7</sup> A qualitative discussion has been given in Refs. 4 and 8 for a straight wire. In this case, the radial electric field due to the surface charges and the azimuthal magnetic field give<sup>4,8</sup> a component of the Poynting vector parallel to the wire, which is responsible for the flow of energy from the battery. Electric and magnetic fields and the corresponding Poynting vectors around a current carrying straight wire are described in more detail in the following. Here we show the Poynting vector schematically in Fig. 2 and mention its general features. It is evident from Fig. 2 that the energy comes out of both terminals of the battery, as shown by the Poynting vectors, and then flows along the wire. The size of the vectors becomes smaller and smaller farther along the wire because some energy has already dissipated and the energy flowing down the rest of the wire is smaller. The relation between the dissipated energy and the Poynting vector will be shown explicitly in what follows. Although this relation was discussed by Sommerfeld,<sup>4</sup> he focused only on the electromagnetic field energy entering the wire perpendicular to its surface and did not discuss how much energy flows parallel to the wire. A more complete qualitative analysis is given in Ref. 8, which we discuss next.

Galili and Goihbarg<sup>8</sup> considered a circuit consisting of a battery connected to a resistor, made of a straight wire, by ideal wires of zero resistivity. These wires are at a constant potential and therefore have an electric field that is zero inside the wires and perpendicular to the wires outside. This electric field coupled with the magnetic field outside the wires gives a Poynting vector which is parallel to the wires. Thus energy is transported to the resistor parallel to these wires. In the resistor and around it, there is also a nonzero electric field parallel to the resistor. This field gives a component of the Poynting vector perpendicular to the resistor that takes the energy into the resistor.

On the basis of Ref. 8, the qualitative picture that emerges for the flow of energy parallel to a wire is as follows. For a conducting wire carrying steady current, there is an electric field parallel to the wire inside. The continuity of the parallel

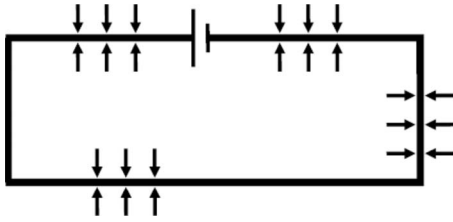


Fig. 1. Poynting vectors as given by Eq. (4) at four sections of a wire connected to a battery.

component of the electric field across a boundary implies that this field continues outside the wire and goes to zero away from it. As a result, because the curl of the electric field must vanish, there has to be a radial component of the field outside the wire. Thus the direction of the field outside a current carrying conductor is at an angle ( $\neq \pi/2$ ) from the normal to the surface of the conductor. The existence of the radial component of electric field outside a current carrying wire was demonstrated experimentally in Ref. 9 and by simulations of a RC circuit in Ref. 10. Such an electric field outside a wire also implies that there are charges on the surface of the wire. The radial electric field and the azimuthal magnetic field around the wire give a component of the Poynting vector parallel to the wire, which implies that the energy transported by the fields moves parallel to the wire. The energy flow shown in Fig. 2 is based on this qualitative picture, and is slightly different from Fig. 3 of Ref. 8 because we have taken the resistivity to be nonzero throughout the wire.

We now make this qualitative picture precise by showing that the Poynting vector given by the radial electric field and the azimuthal magnetic field carries exactly the energy required for Joule heating in the wire. We also show that a large fraction of the energy is transported in regions close to the wire.

In the following, we calculate the electric field outside a long current carrying wire field as discussed in Refs. 4 and 9–15. Consider a long uniform wire of length  $L$  and radius  $a$  on the  $z$ -axis, as shown in Fig. 3. Its upper end is connected to the positive terminal of a battery of  $\text{emf} = V$ . The wire is inside a thin hollow grounded cylinder of radius  $R$  ( $R > a$ ) made of a conductor of zero resistivity. The cylinder is closed at its lower end where the wire is connected to it. The negative terminal of the battery is connected to the cylinder. The electric field inside the conductor is  $\mathbf{E} = -(V/L)\hat{z}$ , and the potential  $\phi(s, z)$  on the surface of the wire varies with  $z$  as  $\phi(s = a, z) = V(z/L)$ . To calculate the electric field between the

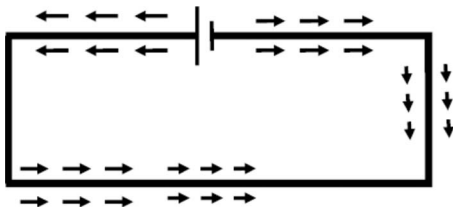


Fig. 2. Component of the Poynting vector parallel to the wire near its surface. This component arises due to surface charges on the wire.

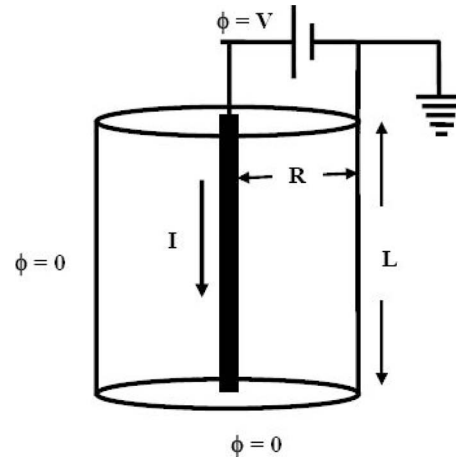


Fig. 3. A wire along the  $z$ -axis connected to a battery of  $\text{emf} V$ . The wire is enclosed in a cylindrical shell that provides the return path for the current.

wire and the cylinder, we consider the wire to be long enough so that fringe effects at the upper end of the cylinder can be neglected. Because there is azimuthal symmetry, the electrostatic potential  $\phi(s, z)$  depends only on  $s$  and  $z$  and satisfies the boundary conditions

$$\phi(s = a, z) = V \frac{z}{L}, \quad \phi(s = R, z) = 0. \quad (6)$$

The solution for the potential is straightforward<sup>9,11,13</sup> and is given by

$$\phi(s, z) = \left( \frac{V}{L} \right) \frac{\ln(R/s)}{\ln(R/a)} z. \quad (7)$$

The corresponding electric field is

$$\begin{aligned} \mathbf{E}(s, z) &= -\frac{\partial \phi}{\partial s} \hat{s} - \frac{\partial \phi}{\partial z} \hat{z} \\ &= \left( \frac{V}{L} \right) \frac{1}{\ln(R/a)} \frac{z}{s} \hat{s} - \left( \frac{V}{L} \right) \frac{\ln(R/s)}{\ln(R/a)} \hat{z}. \end{aligned} \quad (8)$$

The field lines are shown in Fig. 4. The calculated electric field pattern is close to the actual field around a conductor in a similar situation, as depicted in Ref. 9, Fig. 6.

For a long wire, the magnetic field  $\mathbf{B}(s, z)$  and the Poynting vector  $\mathbf{S}(s, z)$  around the wire are

$$\mathbf{B}(s, z) = -\frac{\mu_0 I}{2\pi s} \hat{\phi} = -\frac{\mu_0}{2\pi} \left( \frac{V}{L} \right) \frac{\pi a^2 \sigma}{s} \hat{\phi} \quad (9)$$

and

$$\begin{aligned} \mathbf{S}(s, z) &= \frac{1}{\mu_0} \mathbf{E}(s, z) \times \mathbf{B}(s, z) \\ &= -\frac{1}{2} \left( \frac{V}{L} \right)^2 \frac{a^2 \sigma}{\ln(R/a)} \frac{z}{s^2} \hat{z} - \frac{1}{2} \left( \frac{V}{L} \right)^2 \frac{\ln(R/s)}{\ln(R/a)} \frac{a^2 \sigma}{s} \hat{s}. \end{aligned} \quad (10)$$

Thus, the Poynting vector has two components: One parallel to the wire in the same direction as the current and the other going into the wire. The former is responsible for energy

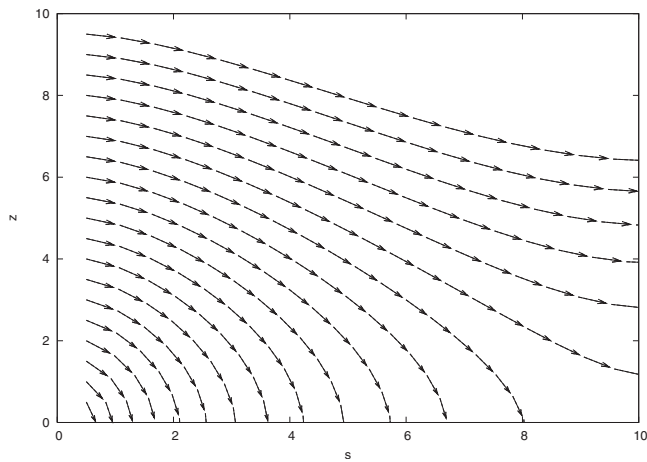


Fig. 4. Electric field lines around a current carrying wire. The parameters used to draw the figure are  $V=10$  V,  $L=10$  m,  $a=0.5$  m, and  $R=10$  m. The arrows are drawn to indicate the direction of the field; their lengths are not proportional to the magnitude of the field.

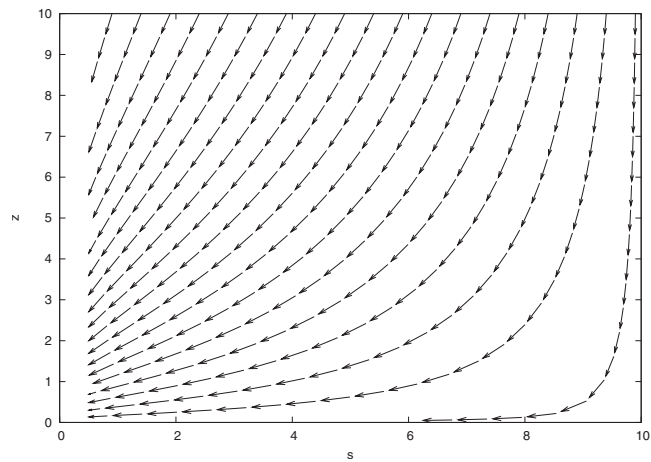


Fig. 5. The Poynting vector around a current carrying wire. The parameters are the same as in Fig. 4. The arrows are drawn to indicate the direction of the Poynting vector, but their length is not proportional to the magnitude of the field. The component parallel to the wire is proportional to  $z$  and therefore becomes smaller as  $z$  increases.

flow along the wire, and the latter takes it into the wire as shown in Fig. 5. It is evident from Fig. 5 that the energy flow near the wire has components both parallel and perpendicular to the wire. As discussed in Ref. 8, the direction of the flow of energy near the outer cylinder is opposite to the direction of the current, although in the present case, its magnitude is miniscule. Note that the magnitude of the component parallel to the wire is proportional to  $z$  and therefore will become smaller as  $z$  approaches 0.

We now calculate the total energy  $E_T$ , which flows across a plane perpendicular to the wire at height  $z$ ,

$$E_T = \int_a^R \frac{1}{2} \left( \frac{V}{L} \right)^2 \frac{a^2 \sigma}{\ln(R/a)} \frac{z}{s^2} 2\pi s ds$$

$$= \sigma \left( \frac{V}{L} \right)^2 \pi a^2 z. \quad (11)$$

It is easy to see that the energy  $E_T$ , which flows parallel to the wire across a plane perpendicular to it at height  $z$ , is equal to the energy that enters the wire between 0 and  $z$  and then appears as Joule heating. To calculate the energy entering the wire between 0 and  $z$ , we substitute  $s=a$  in the radial component of the Poynting vector of Eq. (9) and multiply it by the surface area  $2\pi az$  of the wire between 0 and  $z$ . The result is the same expression as obtained in Eq. (11). Furthermore, the right-hand side of Eq. (11) is the Joule heating in the lower portion of the wire because  $\sigma(V/L)^2$  is the thermal energy produced per unit volume of the wire. This result is independent of the radius  $a$  of the wire as well as the radius  $R$  of the cylinder. Consequently, it holds in the limit as  $R \rightarrow \infty$  and therefore is applicable for a current carrying wire in free space. Thus we have shown that all the thermal energy appearing in a portion of a current carrying conductor reaches there by flowing parallel to the wire.

To verify that a substantial amount of the energy is transported in regions close to the wire, we calculate the radii  $R_{50}$  and  $R_{75}$  of cylindrical regions around a current carrying wire through which 50% and 75% of the total energy  $E_T$  given by

Eq. (11) flow. The energy  $\tilde{E}$  flowing across a circular plane of radius  $\tilde{R}$  with its center on the axis of the wire and at height  $z$  is given by

$$\tilde{E} = \int_a^{\tilde{R}} \frac{1}{2} \left( \frac{V}{L} \right)^2 \frac{a^2 \sigma}{\ln(R/a)} \frac{z}{s^2} 2\pi s ds$$

$$= \sigma \left( \frac{V}{L} \right)^2 \pi a^2 z \frac{\ln(\tilde{R}/a)}{\ln(R/a)}$$

$$= \frac{\ln(\tilde{R}/a)}{\ln(R/a)} E_T. \quad (12)$$

From Eq. (12), we obtain

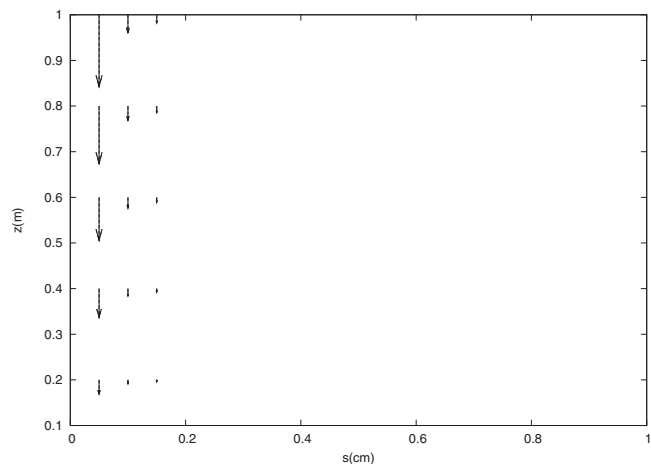


Fig. 6. Poynting vector around a current carrying wire. Parameters used are  $a=1$  mm,  $R=10$  cm,  $L=1$  m, and  $V=10$  V, corresponding to a typical laboratory setup (Ref. 16). Arrows drawn represent the magnitude of the  $z$ -component (parallel to the wire) of the Poynting vector. This component is proportional to  $z$  and becomes smaller as  $z$  increases. The maximum value of  $s$  in the figure is 1 cm, which is ten times smaller than  $R$ .

$$R_{50} = \sqrt{aR} \quad \text{and} \quad R_{75} = R \left( \frac{a}{R} \right)^{1/4}. \quad (13)$$

In a typical laboratory setup,<sup>16</sup> we consider a 1 m long current carrying wire of radius  $a=1$  mm and take  $R=10$  cm. In that case  $R_{50}=1$  cm and  $R_{75}=3.2$  cm. These numbers show that a large fraction of the total energy flows in regions close to the wire. In Fig. 6 we have drawn the  $z$ -component (which is parallel to the current) of the Poynting vector around the wire as a function of  $s$  at five values of  $z$ . It is evident from Fig. 6 that this component of the Poynting vector is largest at the surface of the wire and decreases quickly away from the surface. Also the magnitude becomes smaller as  $z$  approaches zero.

To conclude, we have validated the analysis of Ref. 8 and made it quantitative. Our results show that the energy flow from a battery connected to a long straight wire is along the wire carrying the current. Our results will hopefully prompt similar studies for other geometries of current carrying wires.

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## Curiosity and Critical Thinking

Curiosity is essential but not sufficient. It must be accompanied by a capacity for critical thinking. Unlike curiosity, the critical faculty is not innate. It must be learned and constantly practiced.

George A. Cowan, *Manhattan Project to the Santa Fe Institute* (University of New Mexico Press, 2010) p. 159.